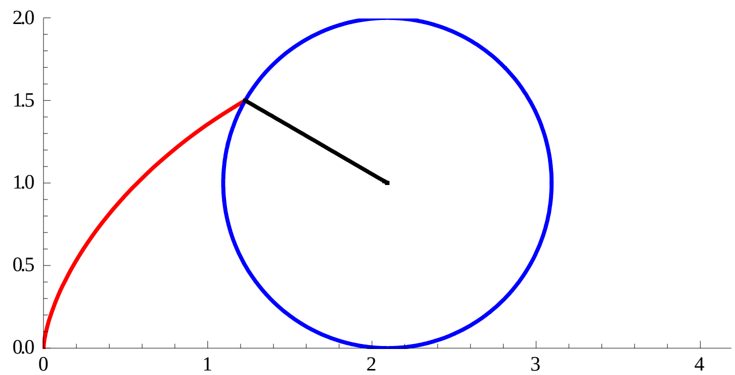
NAME:

Math 250 Practice Exam 2 (Answer Key)

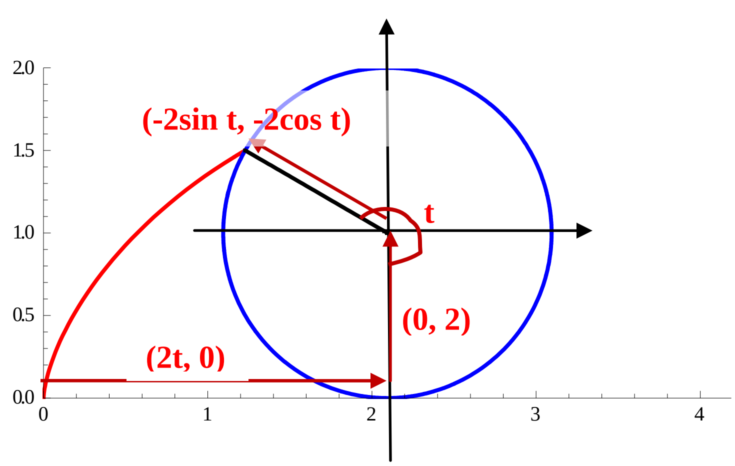
Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. A **cycloid** is a curve that is traced by a point on a rolling circle that travels without slipping along the x-axis.



Find a parametric equation of the cycloid curve when the radius of the rolling circle is 2. [10 pts]

Solution: Attach a gyroscopic coordinate system at the center of the circle. With respect to that system, the circle rotates clockwise, starting from the "negative y-axis" of the gyroscopic coordinate system.



The position of the point that is tracing the curve is the sum of the displacement vectors to the origin of the gyroscopic coordinate system (translation part of motion) and the rotation vector (that rotates clockwise by the angle t starting at $3\pi/2$). Because the circle is rolling without slipping

$p(t) = (2t, 0) + (0, 2) + (2\cos(3\pi/2 - t), 2\sin(3\pi/2 - t))$ which we can simplify to $p(t) = (2t, 0) + (0, 2) - (2\sin t, 2\cos t)$.

2. Let $p(t) = (t, \cos t, e^{2t})$.

(a) Compute $p'(0)$ [2 pts]

Solution: $p'(0) = (1,\ 0, 2)$

(b) Compute $Dp(0)$ [4 pts]

Solution: $Dp(0)(t) = (t, 0, 2t)$

(c) If $p(t)$ represents the position of a particle at time t, what is the physical interpretation of your calculations in (a) and in (b)? [4 pts]

Solution: $Dp(0)(t) = (t, 0, 2t)$ represents the position of the particle, originally travelling along the path $p(t) = (t, \cos t, e^{2t})$, t-units of time after detaching from this path. After leaving the path at $s = 0$, the particle continues to travel in a straight line that is tangent to $p(t)$ with velocity $p'(0)$.

3. Let $k(x, y, z) = (x^2 - y^2 + z, 3x - y - z)$. Compute $Dk(1, 0, -1)$ [10 pts]

Solution: $Jk(x, y, z) = \begin{pmatrix} 2x & -2y & 1 \\ 3 & -1 & -1 \end{pmatrix}$.

Therefore $Jk(1,0,-1) = \begin{pmatrix} 2 & 0 & 1 \\ 3 & -1 & -1 \end{pmatrix}$

and

$Dk(1, 0, -1)(x, y, z) = (2x - z, 3x - y - z)$.

4. Let $f(x, y) = \int_{\pi}^{xy} \frac{\sin t}{t}\, dt$. Compute $Df(\pi/2, 1)$. [10 pts]

Solution: $Jf(x, y) = \begin{pmatrix} \frac{\sin xy}{xy} y & \frac{\sin xy}{xy} x \end{pmatrix} = \begin{pmatrix} \frac{\sin xy}{x} & \frac{\sin xy}{y} \end{pmatrix}$

Therefore $Jf(\pi/2, 1) = (2/\pi \quad 1)$ and $Df\left(\frac{\pi}{2}, 1\right)(x, y) = \frac{2}{\pi} x + y$

5. Let $f(x,y) = x^2 + \cos y$. Find the equation of the tangent plane to the graph $z = f(x,y)$ at the point $(1, \pi/2)$. [10 pts]

Solution: $f\left(1, \frac{\pi}{2}\right) = 1$ and $Df\left(1, \frac{\pi}{2}\right)(x,y) = 2x - y$.
Therefore the equation of the tangent plane is $z = 1 + 2(x-1) - (y - \frac{\pi}{2})$.

6. Let $g(x,y) = x^2 + xy$, $p(s,t) = (s^2 - t, s^2 + t)$, and set $u = g \circ p$. Find $\frac{\partial u}{\partial s}(1,1)$. [10 pts]

Solution: $\frac{\partial u}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} = (2x + y)2s + (x)2s$.
Therefore,

$\frac{\partial u}{\partial s}(1,1) = 4$.

7. Define $u = g \circ f$ where $f(x,y) = (3x + y, -2x + 3y, x + y)$ and $g(x,y,z) = x^2y + z$. Compute $Du(1,0)$. [10 pts]

Solution: Observe that f is linear. Therefore $Jf(1,0) = \begin{pmatrix} 3 & 1 \\ -2 & 3 \\ 1 & 1 \end{pmatrix}$.
$Jg(x,y,z) = (2xy \quad x^2 \quad 1)$

and therefore

$Jg(f(1,0)) = Jg(3,-2,1) = (-12 \quad 9 \quad 1)$.

Now $(-12 \quad 9 \quad 1)\begin{pmatrix} 3 & 1 \\ -2 & 3 \\ 1 & 1 \end{pmatrix} = (-53 \quad 16)$. Thus
$Du(1,0)(x,y) = -53x + 16y$.

8. Let $f(x,y) = x^2y + \sin(\pi xy)$ and let $\vec{u}$ be a direction vector parallel to $\boldsymbol{i} + \boldsymbol{j}$. Calculate the directional derivative $D_{\vec{u}}f(1,-1)$ [10 pts]

Solution: $\nabla f(x,y) = (2xy + \pi y \cos \pi xy,\ x^2 + \pi x \cos \pi xy)$. Therefore

$\nabla f(1,-1) = (-2 + \pi, 1 - \pi)$ and $D_{\vec{u}}f(1,-1) = (-2 + \pi, 1 - \pi) \cdot \frac{1}{\sqrt{2}}(1,1) = \frac{-2+\pi+1-\pi}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$.

9. A thrill-seeking family took their 87 year-old grandma on a hiking trip to the national paraboloid hill whose landscape is the graph $z = 5 - x^2 - y^2$ (in miles). If her current position is $(2, 0, 1)$, in what direction(s) should the poor old grandma head to avoid further climbing (i.e. avoid changing altitude)? [10 pts]

Solution: $\nabla \mathrm{z}(\mathrm{x}, \mathrm{y}) = (-2\mathrm{x}, -2\mathrm{y})$ and hence $\nabla \mathrm{z}(2, 0) = (-4, 0)$. To avoid changing altitude, the grandma must move in a perpendicular direction to the gradient. Therefore she has a choice between $\vec{u}_1 = (0, 1)$ and $\vec{u}_2 = (0, -1)$

10. The point $(1, 0, -1)$ is a solution to the equation $x^2 z + ye^z = -1$.

(a) Does the equation define z as the implicit function $z(x, y)$ for x, y in the vicinity of $(1, 0)$? Explain. [4 pts]

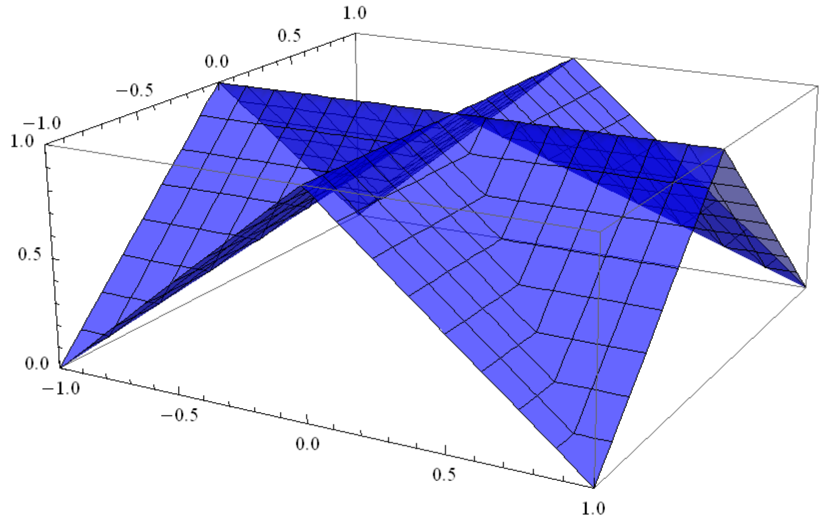
Solution: Let $f(x, y) = x^2 z + ye^z$. Observe that $\frac{\partial f}{\partial z}(1, 0, -1) = 1$ Since this partial derivative isn't 0, it follows from the implicit function theorem that z is defined as an implicit function $z(x, y)$ for x, y in the vicinity of (1, 0).

(b) Compute $\frac{\partial z}{\partial x}$ at $(1, 0, -1)$ [6 pts]

Solution: $\frac{\partial z}{\partial x} = -\frac{\partial f / \partial x}{\partial f / \partial z} = -\frac{2xz}{x^2 + ye^z}$. Therefore, $\frac{\partial z}{\partial x}(1, 0, -1) = 2$.

Extra-Credit

11. The "Victorian cottage roof" is the graph of the function $f(x, y) = 1 - \min\{|x|, |y|\}$ is shown below:



(a) Using your geometric intuition or using the formula of f, compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$. [2 pts]

Solution: $f(x, 0) = f(0, y) = 1$. Since these curves are constant, it follows that $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

(b) Using part (a) what would be your formula for $Df(0,0)$? [3 pts]

Solution: If the total derivative exists, it must be given by the formula $T(x, y) = 0x + 0y = 0$.

(c) According to your intuition, is f differentiable at (0,0)? Is the function obtained in part (b) the derivative of f at (0, 0)? [5 pts]

Solution: No! As we zoom in on the point (0, 0, 1), the graph never begins to look like a plane. More precisely, observe that directional derivatives along the lines y = x and y = -x do not exist at (0, 0).

12. Use chain rule to derive the expression for quotient rule. In particular, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\vec{a} \in \mathbb{R}^n$ with $g(\vec{a}) \neq 0$, then $D(f/g)(\vec{a})(\vec{x}) = \frac{g(\vec{a})Df(\vec{a})(\vec{x}) - f(\vec{a})Dg(\vec{a})(\vec{x})}{[g(\vec{a})]^2}$. [10 pts]

Solution: Define $Q: \mathbb{R}^2 \to \mathbb{R}$ by $Q(u, v) = {}^{u}/_{v}$ and $H: \mathbb{R}^n \to \mathbb{R}^2$ by $H(\vec{x}) = (f(\vec{x}), g(\vec{x}))$. Then $(f/g) = Q \circ H$. Consequently, $D(f/g)(\vec{a})(\vec{x}) = DQ\big(H(\vec{a})\big)DH(\vec{a})(\vec{x})$.

Observe that $DQ(c, d)(u, v) = \left(\frac{1}{d}\right)u - \left(\frac{c}{d^2}\right)v$. In particular, $DQ\big(H(\vec{a})\big)(u, v) = DQ\big(f(\vec{a}), g(\vec{a})\big)(u, v) = \left(\frac{1}{g(\vec{a})}\right)u - \left(\frac{c}{[g(\vec{a})]^2}\right)v$. The corresponding calculation for DH yields$DH(\vec{a})(\vec{x}) = (Dg(\vec{a})(\vec{x}), (\vec{a})(\vec{x}))$. Upon composing and simplifying we obtain the desired quotient formula.

13. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

(a) Is f continuous at $(0, 0)$? [2 pts]

Solution: The function is continuous at $(0, 0)$. To see this, observe that $0 \leq \left|\frac{x^2 y}{x^2+y^2}\right| \leq \frac{\left(\sqrt{x^2+y^2}\right)^3}{\left(\sqrt{x^2+y^2}\right)^2} = \sqrt{x^2 + y^2} \to 0$ as $(x, y) \to (0, 0)$.

(b) Do all the directional derivatives $D_{\vec{u}}f(0, 0)$ exist at $(0, 0)$? [3 pts]

Solution: Let $\vec{u} = (a, b)$. Then $D_{\vec{u}}f(0, 0) = \lim_{t \to 0} \frac{f(t\vec{u}) - f(\vec{0})}{t} = \lim_{t \to 0} \frac{t^3 a^2 b}{t^3(a^2+b^2)} = \frac{a^2 b}{a^2+b^2}$. Therefore all directional derivatives exist.

(c) Is f differentiable at $(0, 0)$? [5 pts]

Solution: The above calculation shows, in particular, that the partial derivatives of $f(x, y)$at (0, 0) are both 0. Thus, the only possible derivative candidate is the linear map $T(x, y) = 0$. However,

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - T(x, y)}{\|(x, y)\|} = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{\left(\sqrt{x^2 + y^2}\right)^3}$$

and the above limit does not exist (To see this, compute the limit along y = x and along the x-axis).